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Lightlike hypersurfaces of Lorentzian manifolds with distinguished screen

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Abstract

We first study the properties of the lightlike mean curvature on a lightlike hypersurface in a Lorentzian manifold. Then, we show the existence of a large class of lightlike hypersurfaces admitting a distinguished screen and study some of their properties. In particular, we find integrability conditions for distinguished screen distributions and give applications in a spacetime which obeys the null energy condition.

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1. Introduction

It is well known that in a semi-Riemannian manifold there are three causal types of submanifolds: spacelike (Riemannian), timelike (Lorentzian) and lightlike (degenerate), depending on the character of the induced metric on the tangent space. In the third case,

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due to the degeneracy of the metric, basic differences occur between the study of lightlike submanifolds and the classical theory of Riemannian and semi-Riemannian submanifolds (see, for example, [1–3,6,8,11]).

Lightlike submanifolds (in particular, lightlike hypersurfaces) are interesting in general relativity since they produce models of different types of horizons (event horizons, Cauchy's horizons, Kruskal's horizons). The idea that the Universe we live in can be represented as a four-dimensional submanifold embedded in a $(4 + d)$-dimensional spacetime manifold has attracted the attention of many physicists. Higher dimensional semi-Euclidean spaces should provide a theoretical framework in which the fundamental laws of physics may appear to be unified, as in the Kaluza–Klein scheme. Lightlike hypersurfaces are also studied in the theory of electromagnetism (see, for example, [3, Chapter 8], and several others sited therein).

In this paper, we study an $(n + 1)$-dimensional lightlike hypersurface (M, g), $n \geq 2$, of a $(n + 2)$-dimensional Lorentzian manifold $(\bar{M}, \bar{g})$, where g is a degenerate metric on M, induced by the Lorentzian metric $\bar{g}$ of $\bar{M}$. We use the notations and some needed results from [3, Chapter 4]. In the lightlike hypersurface case, basic differences occur mainly due to the fact that the normal vector bundle $TM^{\perp}$ is same as the null tangent bundle along a non-zero differentiable radical distribution $Rad(TM)$ of M, defined by

$$Rad(T_xM) = T_xM^{\perp} = \{\xi_x \in T_x(M) : g(\xi_x, X) = 0, X \in T_x(M)\},$$

where $dim(Rad(TM)) = 1$. There exists a Riemannian screen distribution, denoted $S(TM)$, on M which is complementary to the radical distribution such that we have the orthogonal direct sum

$$TM = TM^{\perp} \oplus S(TM). \tag{1}$$

Throughout this paper, we denote by $\mathcal{F}(M)$ the algebra of differentiable functions on M and $\Gamma(E)$ the $\mathcal{F}(M)$-module of differentiable sections of a vector bundle E over M. The manifolds we consider are supposed to be paracompact, smooth and connected. Therefore, the existence of $S(TM)$ is secured. However, in general, $S(TM)$ is not canonical (thus not unique) and the lightlike geometry depends on its choice. It is, therefore, important to look for a screen distribution with good properties. The objective of this paper is to study a set of distinguished structures, denoted by $(S(TM), \xi)$, on M which are useful for a variety of interesting geometric and or physical problems, where ξ is a global normal null section. We show that there is a large class of lightlike hypersurfaces of Lorentzian manifolds for which there do exist distinguished structures of specific screen distributions and their normal null sections having good properties. We also find integrability conditions for some well-chosen distinguished structures. We first study the properties of the lightlike mean curvature and give a general version of the Raychaudhuri's equation.

The problem of finding methods of construction of invariant normalization of lightlike hypersurfaces was earlier studied in [1] from a different point of view.

2. Preliminaries

For the convenience of readers, we repeat the relevant material from [3] without proofs. From [3, p. 79, Theorem 1.1], we know that for a screen distribution $S(TM)$ on M there exists a unique vector bundle $\mathrm{tr}(TM)$ such that for any non-zero local normal section ξ on $\mathcal{U}$ there exists a unique section N of $\mathrm{tr}(TM)|_{\mathcal{U}}$ satisfying

$$\langle \xi, N\rangle = 1; \qquad \langle N, W\rangle = 0, \quad \text{for all } W \in \Gamma(S(TM)|_{\mathcal{U}}). \tag{2}$$

Then, we have the decomposition

$$T\bar{M}|_M = TM \oplus \mathrm{tr}(TM). \tag{3}$$

From now on, ξ denotes a non-zero local (global in Section 4) section of $Rad(TM)$. Using the decompositions (1) and (3), we obtain the formulas

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + h(X, Y), \\ \nabla_X U &= -A^*_U(X) + \nabla^{*t}_X U, \end{aligned} \tag{4}$$

where $X, Y \in \Gamma(TM)$ and $U \in Rad(TM)$. It is easy to check that ∇ is a torsion-free linear connection on M, h is a $\Gamma(\mathrm{tr}(TM))$-valued symmetric $\mathcal{F}(M)$-bilinear form on $\Gamma(TM)$ and A^*_U is $\Gamma(S(TM))$-valued $\mathcal{F}(M)$-linear operator on $\Gamma(TM)$ and is called the *shape operator* of the screen distribution $S(TM)$. From this formulas we define the *second fundamental form* B_U as follows

$$B_U(X, Y) = \langle A^*_U(X), Y\rangle = \langle h(X, Y), U\rangle. \tag{5}$$

It is important to mention that the second fundamental form B_U on M is independent of the choice of screen distribution. We can also obtain the local versions of these formulas for a pair $\{\xi, N\}$ verifying (2). Thus, from decomposition (3), the local Gauss and Weingarten formulas are given by

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + B(X, Y)N, \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}), \\ \bar{\nabla}_X N &= -A_N X + \tau(X)N, \quad \forall X \in \Gamma(TM|_{\mathcal{U}}), \end{aligned} \tag{6}$$

where B, A_N and ∇ are called the local second fundamental form, the shape operator and the induced linear torsion free connection and τ is a 1-form on $TM|_{\mathcal{U}}$. On the other hand, from the decomposition $TM = S(TM) \oplus Rad(TM)$ we obtain the following local Gauss and Weingarten formulas with respect to $S(TM)$

$$\begin{aligned} \nabla_X W &= \nabla^*_X W + C(X, W)N, \quad X \in \Gamma(TM|_{\mathcal{U}}), \\ \nabla_X \xi &= -A^*_\xi X - \tau(X)\xi, \quad W \in \Gamma(S(TM)|_{\mathcal{U}}), \end{aligned} \tag{7}$$

where C, A^*_ξ and ∇^* are called the local second fundamental form, the local shape operator and the induced connection on $S(TM)$.

Let ∇ be the induced linear connection on M and denote by $\bar{R}$ and R the curvature tensor of $\bar{\nabla}$ and ∇, respectively. From [3, p. 93, Eq. (3.1)], $\bar{R}$ and R are related by

$$\bar{R}(X, Y)Z = R(X, Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\nabla_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z), \tag{8}$$

where $A_V \in \Gamma(\mathrm{tr}(TM))$ denotes the shape operator of the lightlike immersion $M \subset \bar{M}$. From Eq. (8), we obtain the Gauss–Codazzi equations of M.

Theorem 1 ([3]). *Let $(M, S(TM))$ be a lightlike hypersurface of a Lorentzian manifold $\bar{M}$ and consider a pair $\{\xi, N\}$ on $\mathcal{U}$. Then, we have the following equations*

$$\langle \bar{R}(X, Y)Z, \xi\rangle = (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z), \quad (9)$$

$$\langle \bar{R}(X, Y)Z, W\rangle = \langle R(X, Y)Z, W\rangle + B(X, Z)C(Y, W) - B(Y, Z)C(X, W), \quad (10)$$

$$\langle \bar{R}(X, Y)\xi, N\rangle = \langle R(X, Y)\xi, N\rangle = C(Y, A_\xi X) - C(X, A_\xi Y) - 2\,\mathrm{d}\tau(X, Y). \quad (11)$$

We need the following from Kupeli [8]. Consider

$$\widetilde{TM} = \frac{TM}{Rad(TM)}, \qquad \Pi : \Gamma(TM) \longrightarrow \Gamma(\widetilde{TM}) \quad \text{(canonical projection)}$$

Denote $\tilde{X} = \Pi(X)$ and $\tilde{g}(\tilde{X}, \tilde{Y}) = g(X, Y)$. It is easy to prove that the operator $\tilde{A}_U : \Gamma(\widetilde{TM}) \longrightarrow \Gamma(\widetilde{TM})$ defined by $\tilde{A}_U(\tilde{X}) = -(\Pi(\bar{\nabla}_X U))$, where $U \in \Gamma(Rad(TM))$ and $X \in \Gamma(TM)$ is a self-adjoint operator. Moreover, it is known that all Riemannian self-adjoint operators are diagonalizable. Let $\{k_1, \ldots, k_m\}$ be the eigenvalues. If $\tilde{S}_{k_i}$, $1 \leq i \leq m$, is the eigenspace of k_i, then

$$\widetilde{TM} = \tilde{S}_{k_1} \perp \cdots \perp \tilde{S}_{k_m}.$$

3. Lightlike mean curvature and Raychaudhuri's equation

Choose a screen distribution $S(TM)$ and denote by $P_S : \Gamma(TM) \longrightarrow \Gamma(S(TM))$ the corresponding projection. Let $U \in \Gamma(Rad(TM))$ be a normal section on M and consider the operator $A_U = A^*_U|_{S(TM)} : \Gamma(S(TM)) \longrightarrow \Gamma(S(TM))$, where A^*_U is the shape operator defined by the Eq. (4) and $W \in \Gamma(S(TM))$. A_U is called *shape operator* on the distribution $S(TM)$ associated with U, which is a self-adjoint and diagonalizable operator. Let $\{k^*_1, \ldots, k^*_m\}$ be the different eigenvalues on M and $S_{k^*_i}(TM)$, $1 \leq i \leq m$ the eigenspace of k^*_i, respectively. Then,

$$S(TM) = S_{k^*_1}(TM) \perp \cdots \perp S_{k^*_m}(TM),$$

and we can find the following local adapted orthonormal basis of eigenvectors

$$\{E^1_1, \ldots, E^1_{r_1}, E^2_1, \ldots, E^2_{r_2}, \ldots, E^m_1, \ldots, E^m_{r_m}\},$$

where $A_U(E^i_j) = k^*_i E^i_j$, with $1 \leq i \leq m$, $1 \leq j \leq r_i$ and r_i is the dimension of the eigenspace of k^*_i. Consider a map $\tilde{P}_S : \Gamma(\widetilde{TM}) \longrightarrow \Gamma(S(TM))$ defined by $\tilde{P}_S(\tilde{X}) = P_S X$.

Then, $\tilde{P}_S$ is a vector bundle isomorphism, and we have

$$\tilde{P}_S(\tilde{A}_U\tilde{X}) = \tilde{P}_S(-\Pi(\overline{\nabla}_X U)) = P_S(-\overline{\nabla}_X U) = P_S(-\overline{\nabla}_{P_S X} U) = A_U(P_S X).$$

Lemma 1. *Let $(M, S(TM))$ be a lightlike hypersurface of a Lorentzian manifold $\overline{M}$. With the above notations k is an eigenvalue of $\tilde{A}_U$ iff k is an eigenvalue of A_U. Furthermore, $\tilde{X}$ is an eigenvector of $\tilde{A}_U$ associated with k if and only if $P_S X$ is an eigenvector of A_U associated with k.*

Proof. It is an immediate consequence of $\tilde{P}_S$ being an isomorphism.

Therefore, we conclude that the eigenvalues associated with a null section are the same for all screen distributions. Thus, we say that the eigenvalues $\{k_1, \ldots, k_m\}$ of A_U are *the principal curvatures* associated with the null normal section U.

Lemma 2. *Let U and $\hat{U}$ be two normal sections such that $\hat{U} = \alpha U$, then if k is an eigenvalue of A_U, then αk is an eigenvalue of $A_{\hat{U}}$ with the same multiplicity.*

Proof. Let $S(TM)$ be any screen distribution and we consider the shape operator $A_{\hat{U}} : \Gamma(S(TM)) \longrightarrow \Gamma(S(TM))$. From Lemma 1, we know that the eigenvalues respect to $\hat{U}$ do not depend on the screen, so if W is an eigenvector of A_U with respect to k, then

$$A_{\hat{U}}(W) = P_S(-\overline{\nabla}_W \hat{U}) = P_S(-\overline{\nabla}_W(\alpha U)) = P_S(-W(\alpha)U - \alpha\overline{\nabla}_W U) = \alpha k W.$$

Thus, αk is an eigenvalue associated with $\hat{U}$.

Let $\{E^i_j; 1 \le i \le m, 1 \le j \le r_i\}$ be a local orthonormal basis of eigenvectors of $\Gamma(S(TM)|_{\mathcal{U}})$. In order to facilitate the notation and depending on the context, we will also denote it by $\{E_a; 1 \le a \le n\}$ and so $A_U(E_a) = k_a E_a$, where k_a may be repeated. It is well known that the *lightlike mean curvature* $H_U : M \longrightarrow \mathbb{R}$ with respect to a normal section U is given by

$$H_U = -\sum_{a=1}^{n} B(E_a, E_a) = -\sum_{a=1}^{n} \langle A_U(E_a), E_a\rangle.$$

It is easy to show that H_U does not depend on both the screen distribution and the orthonormal basis, and so $H_U = -\sum_{a=1}^{n} k_a$.

One of the good properties of the mean curvature is that it does not depend on the screen distribution chosen, but only of the local normal null section U. The geometric objects, in particular the smooth functions, defined on a lightlike hypersurface M often depend on the choice of its structure $(S(TM), \xi)$ and sometimes this fact causes difficulty if the distribution screen is changed. This is why our aim is to look for a good choice of a structure $(S(TM), \xi)$ (see Section 4).

A local null normal section ξ is called *geodesic* if $\overline{\nabla}_\xi \xi = 0$ for which the integral curves of ξ are called the *null geodesic generators*. This condition has interesting geometric and physical meanings and also helps in simplifying the computations. If U is a null normal section on M, then for all $p \in M$ we can scale the null normal section U to be geodesic on a suitable neighborhood $\mathcal{U}$ of p. Let us suppose that the local normal section ξ is geodesic, that is, $\overline{\nabla}_\xi \xi = 0$ on $\mathcal{U}$. We will denote the shape operator A^*_ξ as A_ξ on $\Gamma(TM|_{\mathcal{U}})$ (note that for simplicity we are making an abuse of notation, but this should not cause confusion). Consider the *tidal force operator* $\overline{R}_\xi : \Gamma(TM|_{\mathcal{U}}) \longrightarrow \Gamma(TM|_{\mathcal{U}})$ (see [10, p. 219]) defined as follows

$$\overline{R}_\xi(X) = \overline{R}(X, \xi)\xi = \overline{\nabla}_{[\xi, X]}\xi - \overline{\nabla}_\xi \overline{\nabla}_X \xi.$$

This is a linear and self-adjoint operator, and $\operatorname{trace}(\overline{R}_\xi) = \overline{\operatorname{Ric}}(\xi, \xi)$. We can define $R_\xi : \Gamma(TM|_{\mathcal{U}}) \longrightarrow \Gamma(TM|_{\mathcal{U}})$ in the same way that $\overline{R}$ but using ∇ instead of $\overline{\nabla}$. From (8), it is very easy to show that eventually $\overline{R}_\xi = R_\xi$ and, therefore, we can define the tidal force by means of geometric objects of the lightlike hypersurface.

Proposition 1. *Let ξ be a local null geodesic normal section, then the tidal force operator R_ξ satisfies the equation*

$$\langle R_\xi(X), Y\rangle = \langle -A^2_\xi(X) + (\overline{\nabla}_\xi A_\xi)(X), Y\rangle, \tag{12}$$

where $X, Y \in \Gamma(TM|_{\mathcal{U}})$ and

$$(\overline{\nabla}_\xi A_\xi)(X) = \overline{\nabla}_\xi(A_\xi X) - A_\xi(\overline{\nabla}_\xi X).$$

Proof. This is shown by the following easy computation

$$\begin{aligned}\langle R_\xi(X), Y\rangle &= \langle \overline{\nabla}_{[\xi, X]}\xi - \overline{\nabla}_\xi \overline{\nabla}_X \xi, Y\rangle = \langle -A_\xi([\xi, X]) + \overline{\nabla}_\xi(A_\xi X), Y\rangle \\ &= \langle -A_\xi(\overline{\nabla}_\xi X) + A_\xi(\overline{\nabla}_X \xi) - \overline{\nabla}_\xi(A_\xi X), Y\rangle \\ &= \langle -A_\xi(A_\xi(X)) + (\overline{\nabla} A_\xi)(X, \xi), Y\rangle.\end{aligned}$$

As a consequence of Proposition 1, we state the following important result.

Proposition 2. *Let M be a lightlike hypersurface in a Lorentzian manifold $\overline{M}$. Let ξ be a geodesic normal section on $\mathcal{U}$ and H_ξ be the lightlike mean curvature associated with ξ. Then, we have*

$$\xi(H_\xi) = -\operatorname{Ric}(\xi, \xi) - \operatorname{trace}(A^2_\xi) = -\operatorname{Ric}(\xi, \xi) - \sum_a k^2_a. \tag{13}$$

Proof. Let $\{E_1, \dots, E_n\}$ be a basis of eigenvectors on $S(TM)|_{\mathcal{U}}$, then we can compute the terms of Eq. (12) when $X = Y = E_a$ and we obtain

$$\begin{aligned}\langle(\overline{\nabla}_\xi A_\xi)(E_a), E_a\rangle &= \langle \overline{\nabla}_\xi(A_\xi E_a) - A_\xi(\overline{\nabla}_\xi E_a), E_a\rangle \\ &= \langle \overline{\nabla}_\xi(k_a E_a), E_a\rangle - \langle A_\xi(\overline{\nabla}_\xi E_a), E_a\rangle = \xi(k_a) - \langle A_\xi(\overline{\nabla}_\xi E_a), E_a\rangle \\ &= \xi(k_a) - \langle \overline{\nabla}_\xi E_a, A_\xi(E_a)\rangle = \xi(k_a).\end{aligned}$$

It clear that $\langle A^2_\xi(E_a), E_a\rangle = k^2_a$. Taking the trace in Eq. (12) finally we obtain the desired result.

Now consider the flux of ξ as a local congruence of null geodesic curves. It is known that the *vorticity tensor* ω is the antisymmetric part of $-A_\xi$ and the *shear tensor* σ is the trace-free of the symmetric part of $-A_\xi$. Since A_ξ is symmetric, $\omega = 0$ and

$$\sigma = -A_\xi - \frac{H}{n} I,$$

where I is the identity operator. From Eq. (13), we obtain a version of the vorticity-free Raychaudhuri's equation for lightlike hypersurfaces

$$\xi(H_\xi) = -\mathrm{Ric}(\xi, \xi) - \mathrm{trace}(\sigma^2) - \frac{H^2}{n}. \tag{14}$$

Let $\gamma(t) \subset \mathcal{U}$ be a null generator of $M \subset \overline{M}$, where $\overline{M}$ is a physical spacetime. Then, the null mean curvature restricted to γ is the expansion $\theta(t) = H_\xi(\gamma(t))$ and Eq. (14) restricted to each null generator is the well-known classic Raychaudhuri's equation for a null geodesic (see [5, p. 60] and [7]). This equation shows how the Ricci curvature influences on the deviation of null geodesics of M.

We say that M is *totally geodesic* if the shape operator A_ξ vanishes identically for one (and so for all) null normal section ξ on each neighborhood $\mathcal{U} \subset M$. Related to this definition we recall the following geometric properties of totally geodesic lightlike hypersurfaces. Finally, we also give a more general version of the Raychaudhuri's equation for non-geodesic null normal section.

Proposition 3. *Let M be a lightlike hypersurface of a Lorentzian manifold $\overline{M}$, and $U \in \Gamma(Rad(TM))$ a normal section with $\overline{\nabla}_U U = -\rho U$. Then, we have the formula*

$$U(H_U) = -\mathrm{Ric}(U, U) - \mathrm{trace}(\sigma^2) - \frac{H_U^2}{n} - \rho H_U.$$

Proof. For each point $p \in M$, there exist a neighborhood $\mathcal{U}$ of p and a smooth function $\alpha : \mathcal{U} \longrightarrow \mathbb{R}$ such that $U = \alpha\xi$, where ξ is a geodesic normal section on $\mathcal{U}$. Furthermore,

$$-\rho U = \overline{\nabla}_U U = \overline{\nabla}_{\alpha\xi}(\alpha\xi) = \alpha\xi(\alpha)\xi = \xi(\alpha)U,$$

and so $\rho = -\xi(\alpha)$. Thus, using Lemma 2 and Eq. (13) we deduce

$$\begin{aligned} U(H_U) &= \alpha\xi(\alpha H_\xi) = \alpha\xi(\alpha)H_\xi + \alpha^2\xi(H_\xi) = \alpha\xi(\alpha)H_\xi + \alpha^2(-\mathrm{Ric}(\xi, \xi) - \mathrm{trace}(A_\xi^2)) \\ &= -\rho H_U - \mathrm{Ric}(U, U) - \mathrm{trace}(A_U^2). \end{aligned}$$

Observe that this computation is independent of the chosen neighborhood, and so we have the desired result.

Theorem 2 ([3], p. 88). *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then, the following assertions are equivalent*:

(i) *M is totally geodesic*.
(ii) *There exists a unique torsion-free metric connection* ∇ *induced by* $\overline{\nabla}$ *on M*.
(iii) $TM^{\perp}$ *is a parallel distribution with respect to* ∇.
(iv) $TM^{\perp}$ *is a Killing distribution on M*.

Another equivalence as immediate consequence of the formula (13) is the following result (see [9]).

Theorem 3. *Let M be a lightlike hypersurface in a spacetime* $\overline{M}$ *satisfying the condition* $\mathrm{Ric}(\xi, \xi) \geq 0$ *for every* $\xi \in Rad(TM)$ *on each* $\mathcal{U} \subset M$. *Then, M is totally geodesic iff* $H_\xi = 0$ *for all local sections* $\xi \in Rad(TM|_{\mathcal{U}})$, *that is, M is minimal.*

Observe that physical spacetimes which obey the null energy condition $\overline{\mathrm{Ric}}(\eta, \eta) \geq 0$ for all null vector η on $\overline{M}$ are examples of spacetimes for which is satisfied the assumption of the above theorem. In [6] are obtained some characterization results for totally geodesic (and so minimal) submanifolds in Lorentzian space forms.

Example 2. Consider the well-known *pp*-waves metric defined by

$$\mathrm{d}s^2 = -2f(u, x, y)\,\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}x^2 + \mathrm{d}y^2,$$

on a four-dimensional spacetime manifold $\overline{M}$ where u, v are retarded/advanced timelike coordinates. The null rays are given by $\{u, x, y \text{ constants}\}$. Lightlike hypersurfaces are generated by $\{u = \text{constant}\}$ and the spacelike 2-surfaces are called wave surfaces. Since they are flat, the waves are plane fronted. This metric admits a covariant constant null Killing vector field ξ such that

$$\xi = \partial_v, \qquad \eta_a = -u_{;a}, \qquad \eta_{a;b} = 0, \qquad \eta_a = g_{ab}\xi^b.$$

Thus, the rays are non-twisting, expansion free, shearing free and hence parallel. Consequently, all the Lightlike hypersurfaces, of $\overline{M}$ are totally geodesic.

Eq. (14) gives us a formula for $\xi(H_\xi)$. Now we present a formula of $W(H_\xi)$ for a section W in $S(TM)|_{\mathcal{U}}$ when the 1-form τ vanishes on $\mathcal{U} \subset M$.

Theorem 4. *Let* $(M, S(TM))$ *be a lightlike hypersurface of a Lorentzian manifold* $\overline{M}$ *and* ξ *a normal section such that the* 1*-form* τ *vanishes on* $\mathcal{U} \subset M$. *Then, if W is a section of* $S(TM)|_{\mathcal{U}}$ *we have the formula*

$$W(H) = -\mathrm{Ric}(W, \xi) - \mathrm{div}(A_\xi)(W),$$

where $H = H_\xi$ *and* $\mathrm{div}(A_\xi)(W) = \mathrm{trace}\{X \to (\nabla_X A_\xi)(W)\}$ *with X in* $\Gamma(TM|_{\mathcal{U}})$.

Proof. Let $\{E_a; 1 \leq a \leq n\}$ be a local orthonormal basis of eigenvectors of $\Gamma(S(TM)|_{\mathcal{U}})$. As $W(H) = -\sum_{a=1}^{n} W(k_a)$. We first compute $W(k_a)$ using the Gauss–Codazzi equations.

The formula (9) with $X = W$, $Y = Z = E_a$ is

$$\langle \bar{R}(W, E_a)E_a, \xi\rangle = -(\nabla_W B)(E_a, E_a) + (\nabla_{E_a} B)(W, E_a) - \tau(W)B(E_a, E_a) + \tau(E_a)B(W, E_a).$$

We display the terms of this equation using (10) and the symmetries of the involved operators as follows

$$\begin{aligned}
&\langle \bar{R}(W, E_a)E_a, \xi\rangle = -\langle \bar{R}(W, E_a)\xi, E_a\rangle = -\langle R(W, E_a)\xi, E_a\rangle,\\
&(\nabla_W B)(E_a, E_a) = W(B(E_a, E_a)) - 2B(\nabla_W E_a, E_a) = W(k_a),\\
&(\nabla_{E_a} B)(W, E_a) = \langle \nabla_{E_a}(A_\xi W), E_a\rangle - \langle A_\xi(\nabla_{E_a} W), E_a\rangle = \langle (\nabla_{E_a} A_\xi)(W), E_a\rangle.
\end{aligned}$$

Then, we have

$$W(k_a) = \langle R(W, E_a)\xi, E_a\rangle + \langle (\nabla_{E_a} A_\xi)(W), E_a\rangle - k_a\tau(W) + \langle A_\xi W, \tau(E_a)E_a\rangle.$$

Accordingly,

$$W(H) = -\sum_{a=1}^{n}\langle R(W, E_a)\xi, E_a\rangle - \sum_{a=1}^{n}\langle (\nabla_{E_a} A_\xi)(W), E_a\rangle - H\tau(W) - \left\langle A_\xi W, \sum_{a=1}^{n}\tau(E_a)E_a\right\rangle.$$

From (11), we have that

$$\begin{aligned}
&\mathrm{Ric}(W, \xi) = \textstyle\sum_{a=1}^{n}\langle R(E_a, W)\xi, E_a\rangle + \langle R(\xi, W)\xi, N\rangle\\
&\quad = \textstyle\sum_{a=1}^{n}\langle R(E_a, W)\xi, E_a\rangle - \langle A_\xi W, A_N\xi\rangle - 2\,\mathrm{d}\tau(\xi, W),\\
&\mathrm{div}(A_\xi)(W) = \textstyle\sum_{a=1}^{n}\langle (\nabla_{E_a} A_\xi)(W), E_a\rangle + \langle (\nabla_\xi A_\xi)(W), N\rangle\\
&\quad = \textstyle\sum_{a=1}^{n}\langle (\nabla_{E_a} A_\xi)(W), E_a\rangle + \langle A_\xi(W), A_N\xi\rangle,
\end{aligned}$$

and hence we conclude that

$$W(H) = -\mathrm{Ric}(W, \xi) - \mathrm{div}(A_\xi)(W) - \tau(W)H - 2\,\mathrm{d}\tau(W, \xi) - \left\langle A_\xi W, \sum_{a=1}^{n}\tau(E_a)E_a\right\rangle.$$

Finally, the assumption $\tau = 0$ on $\mathcal{U}$ establishes the formula.

4. Distinguished structures

This section is devoted to study lightlike hypersurfaces admitting a global null normal section ξ on M. Before establishing the main result of this section we need to give some definitions and results.

Definition 1. Let M be a lightlike hypersurface of a Lorentzian manifold $\bar{M}$. A pair $(S(TM), \xi)$ is said to be a global structure on M iff ξ is a non-vanishing global null normal (GNN) section on M.

One can find a large class of interesting examples of lightlike hypersurfaces admitting a global structure. Suppose $\bar{M}$ is a time-oriented Lorentzian manifold and η a smooth global timelike vector field on $\bar{M}$. Let M be a lightlike hypersurface in $\bar{M}$. Then, η restricted to M is a global section of $Rad(TM) \oplus \mathrm{tr}(TM)$. Thus, the projection of $\eta|_M$ onto $Rad(TM)$ provides a non-vanishing GNN section on M.

Definition 2. Let M be a lightlike hypersurface of a Lorentzian manifold $\bar{M}$ admitting a GNN section ξ on M. A Riemannian distribution $D(TM)$ of TM is said to be ξ-distinguished if each section W of $D(TM)$ satisfies $\bar{\nabla}_W \xi \in \Gamma(D(TM))$. In particular, if a screen distribution $S(TM)$ is ξ-distinguished the pair $(S(TM), \xi)$ is called a distinguished structure on M.

Let $(S(TM), \xi)$ be a global structure of M. Consider the shape operator $A_\xi : \Gamma(S(TM)) \longrightarrow \Gamma(S(TM))$ globally defined. From Lemma 1, we have that the eigenvalues respect to ξ do not depend on the screen. Thus, we say that the eigenvalues of A_ξ are *the principal curvatures* associated with the GNN section ξ.

Proposition 4. *Let $(M, S(TM))$ be a lightlike hypersurface of a Lorentzian manifold $\bar{M}$ admitting a geodesic GNN section ξ on M. Then, $S(TM)$ is ξ-distinguished if only if the corresponding 1-form τ from (6) vanishes. In such case, the Ricci tensor of the induced connection ∇ is symmetric.*

Proof. Consider a vector field X on M. Then, we have

$$\tau(X) = -\langle \bar{\nabla}_X \xi, N \rangle = \langle \bar{\nabla}_{(P_S(X)+\lambda\xi)} \xi, N \rangle = -\langle \bar{\nabla}_{P_S(X)} \xi, N \rangle.$$

Thus, $\tau = 0$ if and only if $\bar{\nabla}_{P_S(X)} \xi \in \Gamma(S(TM))$. Finally, it follows from [3, p. 99, Theorem 3.2] that the induced Ricci tensor is symmetric.

In fact, Theorem 3.2 in [3] establishes that given a screen distribution $S(TM)$, then the Ricci tensor is symmetric if and only if $\mathrm{d}\tau = 0$ for any chosen GNN section ξ. Moreover, Proposition 3.4 in [3] ensures that for this case there exists a pair $\{\xi, N\}$ on M such that the corresponding 1-form τ vanishes. Bearing in mind Proposition 4, the condition $\mathrm{d}\tau = 0$ allows us to find distinguished structures $(S(TM), \xi)$ with ξ a geodesic GNN section. Now we are trying to find more examples of lightlike hypersurfaces admitting distinguished structures.

Theorem 5. *Let M be a lightlike hypersurface in a Lorentzian manifold $\bar{M}$ admitting a GNN section ξ. Suppose that $\bar{\nabla}_\xi \xi = -\rho\xi$ and $\{k_1, \ldots, k_m\}$ are the principal curvatures associated with ξ* on M. *Then, there exist a unique ξ-distinguished Riemannian distribution $D_\xi(TM)$ such that*:

(i) *If* $\rho \neq k_i$ *for all* $i \in \{1, \dots, m\}$, *we have the decomposition*

$$TM = Rad(TM) \perp D_\xi(TM),$$

and so $(S(TM) = D_\xi(TM), \xi)$ *is a unique distinguished structure.*

(ii) *If* $\rho = k_{i_0}$ *for some* $i_0 \in \{1, \dots, m\}$, *we have the decomposition*

$$TM = Rad(TM) \perp S_\rho(TM) \perp D_\xi(TM),$$

where $S_\rho(TM)$ *is any eigenspace associated with the eigenvalue* $\rho = k_{i_0}$.

Proof. Suppose $S(TM)$ and $\hat{S}(TM)$ are two different screens. Let $S_k(TM) \leq S(TM)$ and $\hat{S}_k(TM) \leq \hat{S}(TM)$ be both vector subbundles associated with the same eigenvalue k. Let $\{E_1, \dots, E_r\}$ be a orthonormal basis of $S_k(TM)$ and construct the set $\{\hat{E}_1, \dots, \hat{E}_r\}$, where $\hat{E}_j = P_{\hat{S}}(E_j)$, $1 \leq j \leq r$. From Lemma 1, we have that $\{\hat{E}_1, \dots, \hat{E}_r\}$ is a orthonormal basis of $\hat{S}_k(TM)$ satisfying the formulas

$$\begin{aligned} &\overline{\nabla}_{E_j}\xi = -kE_j - \tau(E_j)\xi, \\ &\overline{\nabla}_{\hat{E}_j}\xi = -k\hat{E}_j - \hat{\tau}(\hat{E}_j)\xi, \quad 1 \leq j \leq r, \\ &\hat{E}_j = E_j + \mu_j\xi. \end{aligned} \tag{15}$$

We are interested in finding μ_j such that $\hat{\tau}(\hat{E}_j)$ vanishes, so that $\hat{S}_k(TM)$ is a ξ-distinguished Riemannian distribution. Then, we have

$$\begin{aligned} \overline{\nabla}_{\hat{E}_j}\xi &= \overline{\nabla}_{(E_j+\mu_j\xi)}\xi = \overline{\nabla}_{E_j}\xi + \mu_j\overline{\nabla}_\xi\xi = -kE_j - \tau(E_j)\xi - \rho\mu_j\xi, -k\hat{E}_j \\ &= -k(E_j + \mu_j\xi) \end{aligned} \tag{16}$$

and, therefore, $\mu_j(\rho - k) = \tau(E_j)$. Accordingly, we obtain that if $\rho \neq k$, then it is enough to take $\mu_j = \tau(E_j)/(\rho - k)$ and trivially $\hat{S}_k(TM)$ is unique with $\hat{\tau} = 0$ on $\hat{\tau}_k(TM)$. Thus, we have actually proved both statements taking

$$D(TM) = \bigoplus_{k \neq \rho} \hat{S}_k(TM).$$

Note that if $k_i \neq \rho$ for all i, then $(\hat{S}(TM), \xi)$ is a unique distinguished structure.

Observe that Theorem 5 provides a large class of distinguished structures.

Definition 3. Let M be a lightlike hypersurface of a Lorentzian manifold $\overline{M}$. Then, M is said to be *totally non-geodesic* if all principal curvatures associated with every local section $\xi \in \Gamma(Rad(TM)|_{\mathcal{U}})$ are non-zero everywhere.

Sufficient conditions for rescaling a GNN section ξ to be a geodesic vector field are given in [9, Section 4] for lightlike hypersurfaces in spacetimes. We give some interesting results when M admits a geodesic GNN section.

Corollary 1. *Let M be a totally non-geodesic lightlike hypersurface in a Lorentzian manifold $\overline{M}$ admitting a geodesic GNN section ξ. Then, there exist a unique ξ-distinguished screen distribution. In that case, the tidal force R_ξ is $\Gamma(S(TM))$-valued if and only if $A_N\xi = 0$.*

Proof. The existence of a unique ξ-distinguished screen comes from Theorem 5(i). It remains to prove the second statement. Let $\{E_a; 1 \leq a \leq n\}$ be an orthonormal basis of eigenvectors of A_ξ, from (11), we deduce

$$\langle R_\xi(E_a), N\rangle = \langle R(E_a, \xi)\xi, N\rangle = C(\xi, A_\xi(E_a)) = k_a\langle A_N(\xi), E_a\rangle.$$

As $k_a \neq 0$ and $S(TM)$ is non-degenerate, R_ξ is $\Gamma(S(TM))$-valued iff $A_N(\xi) = 0$.

In particular, all totally non-geodesic lightlike hypersurface in a Lorentzian manifold of constant curvature with a geodesic GNN section ξ admit a unique ξ-distinguished structure such that $A_N(\xi) = 0$.

Example 1. Monge hypersurfaces in $\mathbb{R}^{n+2}_1$. Consider a smooth function $F : \Omega \to \mathbb{R}$, where Ω is an open set of $\mathbb{R}^{n+1}$, then

$$M = \{(x^0, \ldots, x^{n+1}) \in \mathbb{R}^{n+2}_1, x^0 = F(x^1, \ldots, x^{n+1})\}$$

is called a Monge hypersurface. It is easy to check that such a hypersurface is lightlike hypersurface if and only if F is a solution of the partial differential equation

$$\sum_{i=1}^{n+1} \partial_i(F)^2 = 1, \tag{17}$$

where $\partial_i = \frac{\partial}{x^i}$. We choose the radical and transversal vector bundles as those that are globally spanned by

$$\xi = \partial_0 + \sum_{i=1}^{n+1} \partial_i(F)\partial_i, \qquad N = \frac{1}{2}\left\{-\partial_0 + \sum_{i=1}^{n+1} \partial_i(F)\partial_i\right\}.$$

The corresponding screen distribution is given by $\{W_1, \ldots, W_n\}$, where

$$W_r = \partial_{n+1}(F)\partial_r - \partial_r(F)\partial_{n+1}, \quad 1 \leq r \leq n.$$

If we derive Eq. (17) with respect to x^j, we obtain that $\sum_{i=1}^{n+1} \partial_i(F)\partial_{ij}(F) = 0$ for all $1 \leq j \leq n+1$ and we get

$$\overline{\nabla}_\xi\xi = \overline{\nabla}_{(\partial_0+\sum_{i=1}^{n+1}\partial_i(F)\partial_i)}\left(\partial_0 + \sum_{i=1}^{n+1} \partial_i(F)\partial_i\right) = \sum_{j=1}^{n+1}\left(\sum_{i=1}^{n+1} \partial_i(F)\partial_{ij}(F)\right)\partial_j = 0,$$

so ξ is a geodesic GNN section. Furthermore, we are going to prove that $S(TM)$ spanned by W_r is ξ-distinguished.

$$\bar{\nabla}_{W_r}\xi = \bar{\nabla}_{(\partial_{n+1}(F)\partial_r-\partial_r(F)\partial_{n+1})}\left(\partial_0+\sum_{i=1}^{n+1}\partial_i(F)\partial_i\right) = \sum_{i=1}^{n+1}(\partial_{n+1}(F)\partial_{ri}(F)-\partial_r(F)\partial_{n+1,i}(F))\partial_i.$$

Bearing in mind that $\partial_s = \partial_{n+1}(F)^{-1}(W_s+\partial_s(F)\partial_{n+1})$ for all $1 \le s \le n$, we have

$$\begin{aligned}\bar{\nabla}_{W_r}\xi &= \sum_{s=1}^{n}(\partial_{rs}(F)-\partial_{n+1}(F)^{-1}\partial_r(F)\partial_{n+1,s}(F))W_s\\ &\quad+\sum_{s=1}^{n}(\partial_{rs}(F)\partial_s(F)-\partial_{n+1}(F)^{-1}\partial_s(F)\partial_r(F)\partial_{n+1,s}(F))\partial_{n+1}\\ &\quad+(\partial_{n+1}(F)\partial_{r,n+1}(F)-\partial_r(F)\partial_{n+1,n+1}(F))\partial_{n+1}\\ &= \sum_{s=1}^{n}(\partial_{rs}(F)-\partial_{n+1}(F)^{-1}\partial_r(F)\partial_{n+1,s}(F))W_s, \end{aligned} \tag{18}$$

where we have made use of

$$\sum_{s=1}^{n}(\partial_{rs}(F)\partial_s(F))+\partial_{n+1}(F)\partial_{r,n+1}(F)=0.$$

Therefore, $\bar{\nabla}_{W_r}\xi \in \Gamma(S(TM))$ and so is distinguished. From (18), we can easily compute the null mean curvature H_ξ as follows

$$\begin{aligned}H_\xi &= \sum_{r=1}^{n}(\partial_{rr}(F)-\partial_{n+1}(F)^{-1}\partial_r(F)\partial_{n+1,r}(F))\\ &= \sum_{r=1}^{n}\partial_{rr}(F)+\partial_{n+1}(F)^{-1}\partial_{n+1}(F)\partial_{n+1,n+1}(F)=\sum_{i=1}^{n+1}\partial_{ii}(F).\end{aligned}$$

We list the following properties of Monge hypersurfaces, which will obviously hold for M with a distinguished structure. For proofs see [3, Chapter 4].

(1) On a lightlike Monge hypersurface M of R_1^{n+2}, there exists an integrable screen distribution $S(TN)$.
(2) A lightlike Monge hypersurface M of R_1^{n+2} is a product manifold $M = L \times M'$, where L is an open subset of a null line and M' is Riemannian.
(3) The geometry of a lightlike Monge hypersurface M of R_1^{n+2} essentially reduces to the Riemannian geometry of a leaf M' of $S(TM)$.

Finally, we find integrability conditions of the screen distribution for a class of non-totally geodesic lightlike hypersurfaces with non-zero null mean curvature.

Definition 4. A lightlike hypersurface (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be *totally umbilical* if for every local section $\xi \in \Gamma(Rad(TM)|_{\mathcal{U}})$ there exists a smooth function ρ on $\mathcal{U}$ such that

$$B_\xi(X, Y) = \rho g(X, Y), \quad \forall X, Y \in \Gamma(TM).$$

M is *proper totally umbilical* in $\bar{M}$ if the function ρ is non-zero. Furthermore, this definition is independent of the chosen screen distribution $S(TM)$.

Theorem 6. *Let $(M, S(TM))$ be a lightlike hypersurface of a Lorentzian manifold $\bar{M}$. Then, M is totally umbilical if and only if every section $\xi \in \Gamma(Rad(TM)|_{\mathcal{U}})$ is a conformal killing vector field on $\mathcal{U}$, that is, $\mathcal{L}_\xi g = \Omega g$, with $\Omega = -2\rho$.*

Proof. We obtain the result from a straightforward computation

$$\begin{aligned}\mathcal{L}_\xi g(X, Y) &= \xi\langle X, Y\rangle - \langle \mathcal{L}_\xi X, Y\rangle - \langle X, \mathcal{L}_\xi Y\rangle = -\langle A_\xi X, Y\rangle - \langle X, A_\xi Y\rangle \\ &= -2\langle A_\xi X, Y\rangle. \end{aligned} \tag{19}$$

Then, $\mathcal{L}_\xi g(X, Y) = \Omega g(X, Y)$ iff $\langle 2A_\xi X + \Omega X, Y\rangle = 0$ for all $X, Y \in \Gamma(TM)$ iff $A_\xi X = -(\Omega/2)P_s X$, so as per [3, p. 107, Eq. (5.3)], M is totally umbilical.

We refer [5, Chapters 5 and 6] for a variety of examples of fluid spacetimes, with metric (such as conformal Killing, homothetic and Killing) symmetries. Now we quote the following general result for a proper totally lightlike submanifold.

Theorem 7 ([4]). *Let $(M, g, S(TM))$ be a proper totally umbilical lightlike submanifold of a semi-Riemannian manifold $(\bar{M}(\bar{c}), \bar{g})$ of a constant curvature $\bar{c}$. Then, the induced Ricci tensor on M is symmetric if and only if its screen distribution $S(TM)$ is integrable.*

By Corollary 1, a proper totally umbilical lightlike hypersurface M with a geodesic GNN section ξ admits a unique ξ-distinguished screen. Furthermore, by Proposition 4, M also admits a symmetric Ricci tensor. Then, using above Duggal–Jin theorem we have the following result.

Theorem 8. *Let M be a proper totally umbilical lightlike hypersurface of a Lorentzian manifold $\bar{M}(\bar{c})$ of constant curvature with a geodesic GNN section ξ. Then, there exists a unique integrable ξ-distinguished screen distribution $S(TM)$ on M.*

Now we are going to present a procedure to construct integrable screen distributions in a larger class of lightlike hypersurfaces in Lorentzian manifolds. With this aim in mind we need to recall a basic result.

Lemma 3. *Let c be a value of a smooth function* $f : M \to \mathbb{R}$, *where M is a smooth manifold. If at each point of* $f^{-1}(c) = \{p \in M : f(p) = c\}$ *the* $\mathrm{d}f_p$ *is non-zero, then* $f^{-1}(c)$ *is a submanifold of M, called a level hypersurface of f.*

In fact, if for all $p \in M$, $\mathrm{d}f_p$ is non-zero, then *f* is a submersion onto $f(M)$. It is well known that the set of vectors *X* of *TM* tangent to fibers are called *vertical*, denoted by $\mathcal{V}(TM)$. Moreover, $X \in \Gamma(\mathcal{V}(TM))$ iff $\mathrm{d}f(X) = 0$. In particular, let ξ be a GNN section and consider the lightlike mean curvature $H = H_\xi : M \longrightarrow \mathrm{Im}(H) \subseteq \mathbb{R}$ on *M* associated with ξ defined as the trace of $-A_\xi$, as a submersion function. It is well-known that the points where the null mean curvature is $-\infty$ are focal points (singularities) of the lightlike hypersurfaces. We will not consider such points in our work. The following result ensures that, under suitable conditions, we can always find a set of integrable screen distributions by considering *H* as a submersion.

Theorem 9 (Integrability conditions). *Let* (M, g) *be a lightlike hypersurface of a Lorentzian manifold* $(\bar{M}, \bar{g})$ *admitting a geodesic GNN section* ξ *and satisfying the condition*

$$\mathrm{Ric}(\xi, \xi) \neq -\mathrm{trace}(\sigma^2) - \frac{H_\xi^2}{n} \tag{20}$$

for one (and so for all) geodesic GNN section ξ *on M. Then,* $H = H_\xi : M \longrightarrow \mathrm{Im}(H_\xi) \subseteq \mathbb{R}$ *is a submersion and, therefore, M admits an integrable screen distribution* $S^*(TM) = \mathcal{V}(TM)$.

Proof. We must prove that $\mathrm{d}H|_p \neq 0$ for all $p \in M$. For this, it is enough to prove that there exists a vector v in T_pM such that $\mathrm{d}H|_p(v) \neq 0$. By taking $v = \xi_p$ and using the Raychaudhuri's equation, we get

$$\mathrm{d}H_p(\xi_p) = \xi_p(H) = -\mathrm{Ric}(\xi_p, \xi_p) - \mathrm{trace}(\sigma_p^2) - \frac{H(p)^2}{n}.$$

Therefore, $\mathrm{d}H_p \neq 0$ for all *p* if only if it is satisfied (20). As *H* is a submersion we take $S^*(TM) = calV(TM)$, then $S^*(TM)$ is Riemannian and clearly integrable.

Note that in this case for all *p* in *M*, there is an *n*-dimensional integral submanifold $\Sigma_p = H^{-1}(H(p))$ such that $T_p\Sigma = S_p^*(TM)$. The preceding theorem provides a screen distribution $S^*(TM) = \mathcal{V}(TM)$ characterized by the property $W(H_\xi) = 0$ for all section *W* of $S^*(TM)$.

Corollary 2. *Let* (M, g) *a lightlike hypersurface of a spacetime* $(\bar{M}, \bar{g})$ *admitting a geodesic GNN section* ξ *on M satisfying the null energy condition* $\mathrm{Ric}(\xi, \xi) \geq 0$. *Let* $(S(TM), \xi)$ *be a distinguished structure on M satisfying the condition*

$$\mathrm{Ric}(W, \xi) = \mathrm{div}(A_\xi)(W), \quad \text{for all } W \in \Gamma(S(TM)). \tag{21}$$

Then, $S(TM)$ is integrable.

Proof. We take again the null mean curvature $H \neq 0$ of M associated with the geodesic GNN section ξ. Since $\mathrm{Ric}(\xi, \xi) \geq 0$, Eq. (20) is satisfied, and so H is a submersion. On the other hand, $(S(TM), \xi)$ is distinguished with ξ a geodesic GNN section, so $\tau = 0$ on M, and using Theorem 4, we have that $W(H) = 0$ for all $W \in \Gamma(S(TM))$. The vertical vector fields are characterized by this property, therefore, $S(TM) = \mathcal{V}(TM)$.

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